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Eyespace Values in Go

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ABSTRACT. Most of the application of combinatorial game theory to Go has been focussed on late endgame situations and scoring. However, it is also possible to apply it to any other aspect of the game that involves counting. In particular, life-and-death situations often involve counting eyes. Assuming all surrounding groups are alive, a group that has two or more eyes is alive, and a group that has one eye or less is dead.

This naturally raises the question of which game-theoretical values can occur for an eyemaking game. We define games that provide a theoretical framework in which this question can be asked precisely, and then give the known results to date. For the single-group case, eyespace values include 0, 1, 2, $\int \frac{1}{2}$, $\int 1\frac{1}{2}$, $\int \frac{3}{4}$, $\int 1\frac{1}{4}$, $\int 1*$, and several ko-related loopy games, as well as some seki-related values. The $\int \frac{1}{2}$ eye is well-understood in traditional Go theory, and $\int 1\frac{1}{2}$ only a little less so, but $\int \frac{3}{4}$, $\int 1\frac{1}{4}$, and $\int 1*$ may be new discoveries, even though they occur frequently in actual games.

For a battle between two or more opposed groups, the theory gets more complicated.

1. Go

1.1. Rules of Go. Go is played on a square grid with Black and White stones. The players alternate turns placing a stone on an unoccupied intersection. Once placed, a stone does not move, although it may sometimes be captured and removed from the board.

Stones of the same color that are adjacent along lines of the grid are considered to be connected into a single indivisible *unit*. For example, if one takes the subgraph of the board grid whose vertices are the intersections with Black stones and whose edges are the grid lines that connect two such vertices, the Black units are the connected components of that subgraph. (There is little terminological consistency in the English-language literature; a unit has also variously been called a chain [Remus 1963; Zobrist 1969; Harker 1987; Kraszek 1988], a string [Hsu and Liu 1989; Berlekamp and Wolfe 1994], a group [Thorp and Walden 1964; 1972; Fotland 1986; Becker 1987], a connected group [Millen 1981], a block

[Kierulf et al. 1989; Kierulf 1990; Chen 1990], or an army [Good 1962]. This is unfortunate because the terms chain, group and army are frequently used for completely different concepts.)

An empty intersection adjacent to one or more stones of a unit along a grid line is said to be a *liberty* for that unit. When a player's move reduces a unit to zero liberties, that unit is *captured*, and its stones are removed from the board; the intersections formerly occupied by them revert to being empty. A player may play on any empty intersection, except that the move may not recreate a position that existed at the end of any earlier move (the generalized ko rule), and (in some rules) may not end up part of a unit with no liberties (the no suicide rule). Note that capture of enemy stones happens before one's own liberties are counted, and any capture creates at least one liberty, so that any move that captures is guaranteed not to be a suicide.

For a more detailed discussion of the rules of Go, their variants, and the mathematical formalization thereof, see appendices A and B of [Berlekamp and Wolfe 1994]. There have also been earlier attempts to formalize the rules of Go, including [Thorp and Walden 1964; 1972; Zobrist 1969; Ing 1991].

1.2. Fundamentals of life and death in Go. Single-point eyes. When a unit (or a set of units of the same color) completely surrounds a single empty intersection, we call that intersection a *single-point* eye. One of the most fundamental "theorems" of Go is that it is possible for sets of units to become uncapturable even against an arbitrarily large number of consecutive moves by the opponent. Consider a single Black unit that surrounds two or more single-point eyes (Figure 1, left); even if White removes all other liberties first, it is still illegal by the suicide rule for White to play in either single-point eye. Thus the unit is unconditionally alive, with no further need for Black to play to defend it.

Figure 1. Left: A unit with two single-point eyes cannot be captured. Right: Units may share eyes to make life.

1.3. Static life and topological life. More generally, a set of units can achieve life through shared eyes. We introduce some definitions. A *group* is a set of units of the same color. A group is said to be *alive* if no unit of the group can be captured given optimal defense, even if the opponent moves first. (This definition is more vague and flawed than it might seem at first: to give just one example, it is possible for two disjoint groups to both be alive by this definition, and yet have it be the case that the opponent has a move that threatens both simultaneously so that only one of the two can be saved.) A group is said to be *statically alive*

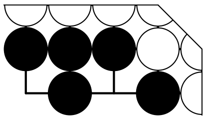


Figure 2. This Black group has one real eye and one false eye, and is dead.

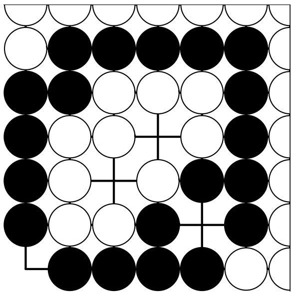
if the opponent cannot capture any unit of the set even given an arbitrarily large number of consecutive moves. Most alive groups are alive because they can achieve static life, although other possibilities (such as seki) exist. A group is said to be *topologically alive* if (1) the units of the group completely surround two or more single-point eyes, and (2) each unit of the group is adjacent to at least two of those eyes. It is straightforward to prove that topologically alive implies statically alive: each unit of the set has at least two liberties from the eyes alone, so an opponent move into any of the eyes captures no units, hence would form a unit with no liberties that dies immediately (such a move is forbidden by the no suicide rule if it applies). The opposite is not true, since a group may be statically alive even if one or more of its eyes are not single-point. However, it appears to be the case (although I will not attempt to prove it here) that any statically alive group can be made into a topologically alive one, and also that any topologically alive group can be made into one with only two eyes. This is the basis for the traditional beginner's guideline that "you need two eyes to live".

For an alternate formulation of life and death fundamentals, which goes beyond that presented here, see [Benson 1976; 1980; Müller 1995, pp. 61–65].

1.4. False eyes. When condition (2) of topological life is violated, the units adjacent to only one eye may be subject to capture (which would destroy the eye) unless they are connected to the group by filling in the eye (which, since the eye is single-point, also destroys it). Such an eye is called a *false eye* in the Go literature, since it does not contribute to life (Figure 2).

The traditional criterion for determining whether an eye is false is local in nature: In addition to occupying all the intersections adjacent to the eye, one must also control enough of the diagonal points (one if the eye is in a corner, two if on an edge, and three if in the middle of the board). Thus the following eyes are all considered false:

This criterion is sometimes inaccurate, since a locally "false" eye may be globally real if the topology of the group includes a loop that contains the eye. This leads to the apparent paradox of "living with two false eyes", which is really not a paradox at all since it clearly falls within the definition of topological life given above. The following Black group is alive even though both its eyes are locally false:



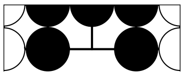
Situations like this are extremely rare, but not unknown, in actual play. Nakayama [1989] comments on one occurrence in a game between Hotta Seiji 4-dan (White) and Nakano Hironari 5-dan (Black) on November 3, 1988. There was also a group that lived with one real and one false eye in a game between Shinohara 9-dan and Ishigure 8-dan some twenty years earlier [Haruyama 1979].

1.5. Eyespaces. An *eyespace* for a group X is a set E of connected intersections such that there exists a sequence of moves that results in at least one intersection of E being an eye of a group X', and the stones of X' include all the stones of X. It sometimes occurs that a group has the potential to make eyes in more than one area, and that moves made in one of these eyespaces do not affect which moves are legal in the other ones. In this case we say that the eyespaces are *independent*. When this happens, the normal theory of sums of games (described in the next section) applies; we can evaluate each area separately, then just add up the results.

To get an approximate measure of the frequency with which multiple separate eyespaces occur in real situations, I examined all the problems in Maeda's tsume-go series [Maeda 1965a; 1965b; 1965c]. This set of books presents 585 life-and-death problems, ranging from elementary to advanced. Sixty-one of them (10.4%) seem to have two independent eyespaces. There are also several problems whose key is that two eyespaces that naively appear to be independent in fact have a subtle interaction that can be exploited (see for example [Maeda 1965b, problems 88, 127, and 163]), and at least one [Maeda 1965b, problem 67] where the key is to make two captures simultaneously so that the opponent can only play in one of the resulting independent eyespaces.

Determining independence can sometimes be quite difficult. However, if we are unable to prove that two eyespaces are independent, we can just lump them together and treat them as a single eyespace. Also, if we merely want to construct an example of a group with multiple independent eyespaces, it is simple to do so by putting solid walls of stones between them. Thus the occasional difficulty of establishing independence does not present a serious impediment to developing a theory of eyespace values.

1.6. Conventions for eyespace diagrams. As in [Berlekamp and Wolfe 1994], we draw diagrams with the convention that White stones cut by the diagram boundary are assumed to be *immortal*, that is, safely connected to a White group with two or more eyes in such a way that no set of moves within the diagram itself can have any effect on their safety. Unlike [Berlekamp and Wolfe 1994], however, Black stones cut by the boundary are not considered to be immortal, but only to be connected out to the remainder of the Black group (which may have other eyespaces) with sufficient liberties that their liberty count does not affect the analysis within the diagram. This condition on liberties is necessary, as can be seen in this example:



As long as the Black unit has at least one liberty outside of the diagram, it is worth one eye for Black; White has no legal move. However, if we maliciously assume that no such liberty exists, then White can capture the Black unit by playing on the empty intersection.

Thus, all the eyespace diagrams should be interpreted as follows: The Black stones cut by the boundary are part of a "backbone" unit that solidly connects all the eyespaces of the group, and that also has enough external liberties that the local analysis need not worry about it being captured.

Normally the backbone will only connect to the diagram at one point. When multiple separate Black units are cut by the boundary, we need to know whether loops exist containing the eyespace (i.e., which of the units is connected to the backbone), and thus whether any eyes that are locally false are also globally false. We assume that all Black units that are contiguous along the cut-line are connected, and only those separated by intervening White stones (as in the false-eye diagrams on page 205) are not.

2. Modeling Life and Death Problems

2.1. The phases of a life and death battle. From a Go player's perspective, life-and-death battles proceed through two or three distinct phases:

- A hot eye-making phase, where Black and White each attempt to make eyes for themselves, and to destroy the other player's eyes. This continues until each group either has at least two eyes (so it lives), or has at most one eye (so it dies), or rarely the situation becomes played out in some other way (seki, bent-four, mannen ko, and so on).
- If both sides live, then comes a warm point-making phase in which the remaining points between the Black and White groups are decided. This phase may include some moves that threaten a group's life, but they will almost always be answered.

- A cool phase (what [Berlekamp and Wolfe 1994, p. 124] calls an *encore*), in which neither player's moves are worth much. In Japanese rules this phase is not even played; in Chinese rules it is part of the final filling in of territory and capturing of dead stones.

Since the difference between a group living and dying is usually large (locally at least fourteen points, and often much more), the first phase can be very hot. Given the current state of game theory, it is difficult (and not very enlightening) to try to analyze a life-and-death battle in terms of points. Our intuition is that the first phase above, the eye-making phase, is the most important. Further, optimal play in an eyespace usually does not depend on the number of points at stake if the group lives or dies. It seems worthwhile to develop a theoretical framework in which only the number of eyes made matters.

2.2. Tsume-go and Bargo. The study of life-and-death problems in Go is called, in Japanese, *tsume-go*; but the term has no direct connotation of life or death. It derives from the transitive verb *tsumeru*, whose main meaning is to stuff, fill, pack, or plug up, and that can also mean to checkmate or to hold one's breath. The sense is that life-and-death problems occur when a group becomes closely surrounded and can no longer run away to safety.

When a Black group is surrounded by White groups that are already alive, and is thus isolated from other Black groups, it must make life on its own. Such a Black group will usually die unless it can make at least two eyes. Although for the purposes of life-and-death we do not care whether Black makes more than two eyes, it is natural to first study the games that may occur without worrying about that limit. To model this, we define the game Bargo, which has the same rules as Go except for scoring; in Bargo, the final score is the number of distinct Black eyes. It doesn't seem to matter much whether we count any kind of eye or only single-point eyes, since any eye should be convertible to a single-point one. The reader demanding mathematical rigor can assume we are counting single-point eyes as defined above.

2.3. Ignoring Infinitesimals. Cooling, chilling, and warming. The $\mathcal{E}$ Operator. This paper presumes that the reader is familiar with the theory of two-person, zero-sum, perfect-information games developed in [Milnor 1953; Hanner 1959; Conway 1976; Berlekamp et al. 1982; Yedwab 1985; Berlekamp 1988; Berlekamp and Wolfe 1994]. In this theory, every move matters, and games are played out until one player has no moves (and thus loses). When considering the eyes of a group, however, we are normally uninterested in whether or not there are moves remaining after the number of eyes has been decided. This leads to some theoretical difficulties: the standard theory treats $\{1|0\}$ as different from $\{1|*\}$ or $\{1*|0\}$, but for our purposes here they are identical.

Roughly, this difference corresponds to the question of allowing or disallowing passes. If passing is allowed, it is impossible to not have a legal move. The precise

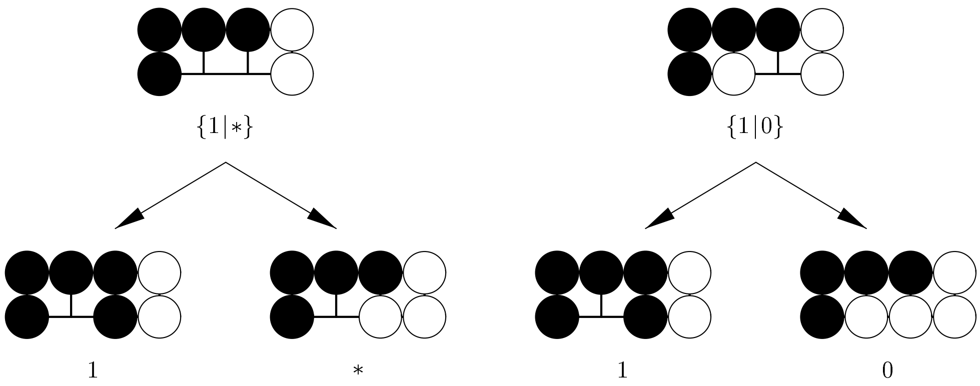


Figure 3. Each of these positions chills to $\{0\,|\,1\} = \frac{1}{2}$.

Go endgame theory requires either forbidding passes, or charging one point for them [Berlekamp and Wolfe 1994, Appendices A and B]. By doing neither, we lose some rigor and exactness. It is not even clear (yet) who "wins" Bargo, since we can no longer define winning as getting the last move.

One approach to this dilemma is to define equivalence classes of games modulo *small* [Conway 1976, p. 100–101] or *infinitesimal* games. That is, two games are in the same equivalence class if their difference is infinitesimal. We define $\langle G\rangle$ to be the class of games H such that $G - H$ is infinitesimal. Most standard operations on games have natural mappings into operations on these classes; $\langle G\rangle + \langle H\rangle = \langle G+H\rangle$, $\mu(\langle G\rangle) = \mu(G)$, etc.

Cooling [Conway 1976, p. 103] by any amount greater than zero eliminates these infinitesimal differences. Cooling by 1, or *chilling*, has an important place in both the endgame theory and here. It is easy to find eye-making games that differ only slightly, and chill to the same game: see Figure 3.

Warming is the approximate inverse of chilling. Both games in Figure 3 are infinitesimally close to $\int \frac{1}{2}$, where $\int$ represents the warming operator of mathematical Go endgame theory [Berlekamp and Wolfe 1994, p. 52 ff.]. We can encompass all such games by defining a one-to-many warming $G \mapsto \left\langle \int G\right\rangle$, and gain further notational convenience by defining a postfix operator $\mathcal{E}$ with dimension of "eyes" such that $G\mathcal{E} = \left\langle \int G\right\rangle$ eyes. Since $\int$ is linear, so is $\mathcal{E}$. As in the endgame theory, integers are unchanged by warming, so that if n is an integer, $n\mathcal{E} = \left\langle \int n\right\rangle$ eyes $= \langle n\rangle$ eyes $= n$-ish eyes. The "ish" suffix can be read as "infinitesimally shifted"; for the purposes of this paper it means "plus or minus an infinitesimal". $\langle G\rangle$ is just the set of all games that are G-ish.

2.4. "Half eye". The situation of Figure 3, where a potential eye can be made by Black (in gote) or permanently destroyed by White (also in gote), is fairly common. This is sometimes called a "half eye" [Davies 1975, p. 71]. In what sense is it really one-half?

If we write this game with the number of Black eyes as the score, it is $\{1|0\}$-ish. A little analysis shows that the sum of two copies of the game always results in exactly one eye, no matter who moves first, as long as the opponent responds. If Black moves first, he makes one eye and White eliminates the other; if White moves first, she eliminates one eye and Black makes the other. This is equivalent to saying that $\{1|0\}+\{1|0\} = 1$. From this it follows that $\mu(\{1|0\}) = \frac{1}{2}$, because

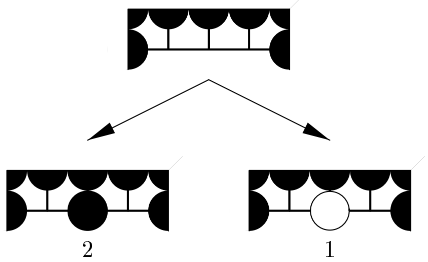
$$2\mu(\{1|0\}) = \mu(\{1|0\})+\mu(\{1|0\}) = \mu(\{1|0\}+\{1|0\}) = \mu(1) = 1.$$

So the mean value of the game is one-half eye. But, more powerfully, we can observe that $\{1|0\}$ chilled is $\{0|1\} = \frac{1}{2}$, and that $\langle \int \frac{1}{2} \rangle = \langle \{1|0\} \rangle$. So we can write the game as simply $\frac{1}{2}\mathcal{E}$ (pronounced "one-half eye"). The power of this notation is great; instead of the detailed case analysis above, it is enough to note that

$$\tfrac{1}{2}\mathcal{E} + \tfrac{1}{2}\mathcal{E} = (\tfrac{1}{2} + \tfrac{1}{2})\mathcal{E} = 1\mathcal{E} = 1\text{ish eye}.$$

This simple formulaic reduction constitutes a rigorous proof that "half an eye plus half an eye equals one eye".

2.5. An eye and a half. Another game that appears fairly frequently is $\{2|1\}$ eyes, which can be written $1\frac{1}{2}\mathcal{E}$. The simplest example of this is a three-point eyespace:



Most Go books refer to this kind of eyespace with terms such as "unsettled shape" [Davies 1975], or merely note that it is "intermediate" between life and death [Segoe 1960]. Such wording is unacceptably vague; as we shall see, there are several different values of eyespaces that could be covered by those descriptions. In addition, a group with an "unsettled" eyespace may in fact be quite settled if it has another eyespace worth half an eye or more. That is, "unsettled" is really best applied to an entire group, and it does not make much sense to apply it to a single eyespace of a group with multiple eyespaces.

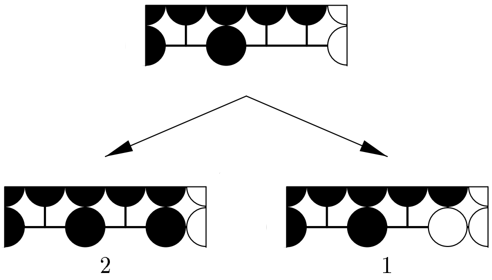
Since games can be translated by adding numbers [Conway 1976, p. 112], we have the equality

$$\{2|1\} = 1+\{1|0\},$$

or, in terms of eyes,

$$1\tfrac{1}{2}\mathcal{E} = 1\,\text{eye} + \tfrac{1}{2}\mathcal{E}.$$

In Go terms, this means that an "eye and a half" situation like that shown above is exactly equivalent (in terms of eyes made) to a single secure eye plus an eye in gote:



2.6. Larger "unsettled" shapes. Restriction to two eyes. Bargo$_{[0,2]}$. Collapsing. When we analyze larger "unsettled" shapes in Bargo, we frequently find games that have one or more integer endpoints of greater than two eyes. For example, the common four- and five-point nakade shapes

have Bargo values of $\{3|1\}$-ish eyes and $\{\{3|2\}|1\}$-ish eyes, respectively. These values can be written $2{*}\mathcal{E}$ and $1\frac{3}{4}\mathcal{E}$. However, for living, there is no value to additional eyes beyond two. To model this effectively, we need to restrict Bargo so that more than two eyes don't count. Since Bargo is also restricted by definition to nonnegative eyes, I call this restricted game Bargo$_{[0,2]}$.

To calculate game values in Bargo$_{[0,2]}$, we first analyze the game as in Bargo (simplifying to canonical form with number-ish stopping points), then restrict the game by changing all stopping points that are greater than 2 to 2 (resimplifying if necessary). This process of restriction I call *collapsing*. We write the game that G collapses into as $c(G)$. Formally:

(i) $c(G) = 2$ if $G \geq 2$, else
(ii) $c(G) = G$ if G is number-ish, else
(iii) $c(\{G^B|G^W\}) = c\{c(G^B)|c(G^W)\}$ if $c(G^B) \neq G^B$ or $C(G^W) \neq G^W$, else
(iv) $c(G) = G$.

Using the above shapes as examples, we see that $\{3|1\}$ collapses to $\{2|1\}$, and $\{\{3|2\}|1\}$ collapses to $\{\{2|2\}|1\} = \{2{*}|1\}$. Generalizing collapsing to apply to the equivalence classes of games defined earlier, we can write $c(2{*}\mathcal{E}) = 1\frac{1}{2}\mathcal{E}$, and also $c(1\frac{3}{4}\mathcal{E}) = 1\frac{1}{2}\mathcal{E}$. So in Bargo$_{[0,2]}$, because of the restriction to two eyes, both these shapes have the same eye value as a three-point eye, although the larger eyeshapes ("big eyes") are worth more liberties. This is in good accord with the way Go players view these shapes. Another way of looking at this is that collapsing induces an equivalence relation $\sim$, where $G \sim H$ if and only if $c(G) = c(H)$; we can say $2{*}\mathcal{E} \sim 1\frac{3}{4}\mathcal{E} \sim 1\frac{1}{2}\mathcal{E}$. In each equivalence class, there is a unique element G for which $c(G) = G$; we say such a G is already *collapsed*. G is the natural representative of the class, since $c(H) = G$ for all H in the class.

Since a collapsed game has no endpoints greater than two, collapsing it again has no effect, which is to say that collapsing is idempotent: $c(c(G)) = c(G)$. For the games occurring in Bargo, which have all nonnegative endpoints, it also has the property that

$$c(G + H) = c(c(G) + c(H)).$$

But it is not always the case that $c(G + H) = c(G) + c(H)$, since $c(G) + c(H)$ may have endpoints greater than 2.

2.7. Semigroup structure. Collapsed addition. Complement. The set $\mathbf{Ug}_{[0,]}$ of games for which all integer-ish endpoints are ≥ 0 is closed under addition, and forms a partially ordered regular abelian semigroup with identity 0. It is different from the set $\mathbf{Ug}^+$ of all games that are nonnegative; neither is a proper subset of the other, since $\downarrow$ is in $\mathbf{Ug}_{[0,]}$ but not $\mathbf{Ug}^+$, while $\int \frac{1}{4}$ is in $\mathbf{Ug}^+$ but not $\mathbf{Ug}_{[0,]}$. The structure of $\mathbf{Ug}_{[0,]}$ appears interesting but is beyond the scope of this paper. The games in Bargo are a proper subset of those in $\mathbf{Ug}_{[0,]}$.

The set $\mathbf{Ug}_{[0,2]}$ of games for which all integer endpoints are 0, 1, or 2 is not closed under normal addition, but is closed under *collapsed addition* $\oplus$, defined by $G \oplus H = c(G + H)$. With this operation, $\mathbf{Ug}_{[0,2]}$ and $\text{Bargo}_{[0,2]}$ each form a partially ordered abelian semigroup with identity 0. Since the games in $\text{Bargo}_{[0,2]}$ are a subset of $\mathbf{Ug}_{[0,2]}$, $\oplus$ seems to be the natural addition operation for $\text{Bargo}_{[0,2]}$. The clipping of values to be ≥ 0 is symmetric with the clipping of values to be ≤ 2, so $\mathbf{Ug}_{[0,2]}$ has a reflective symmetry about 1, that is, the mapping $f(x) = 2 - x$ is a self-inverse isomorphism for $\mathbf{Ug}_{[0,2]}$. For each (collapsed) G, there is a unique H such that $H = f(G)$, $G = f(H)$, and $G + H = 2$. We call H the *complement* of G. Even though neither semigroup can have inverses (negatives) for elements other than 0, the complement acts like an inverse in some ways. In fact, by subtracting 1, we can map from complements in $\mathbf{Ug}_{[0,2]}$ to inverses in $\mathbf{Ug}_{[-1,1]}$, which is an abelian group (under its own, appropriately clipped, addition).

It is not clear whether $\text{Bargo}_{[0,2]}$ has the same reflective symmetry as $\mathbf{Ug}_{[0,2]}$, since we have no proof that if G is a possible eyespace value in Go, then $2 - G$ is also. There may exist elements in $\text{Bargo}_{[0,2]}$ that do not have complements in $\text{Bargo}_{[0,2]}$. The value of "seki" (see next section) is possibly such an element.

3. Examples of Single-Group Values

This section gives examples of most of the known eyespace values for finite games in $\text{Bargo}_{[0,2]}$, as well as of some values for simple loopy games. For each value, one has been worked out in detail, and the others are left as exercises.

3.1. $\frac{1}{2}\mathcal{E}$**.** In each of the examples in Figure 4, Black moving first can make one secure eye, and White moving first can eliminate Black's eye potential. Some of the examples are simple; in others it is more difficult to see how Black or White should play.

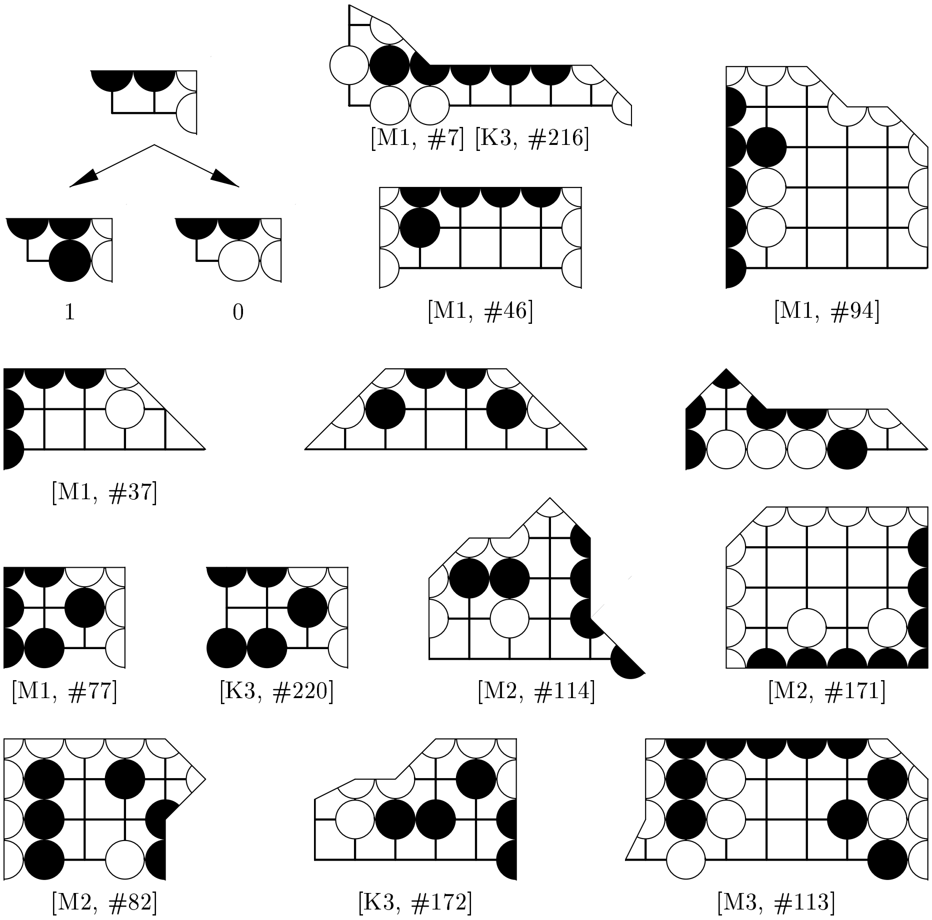


Figure 4. In each example, black moving first can go to a position of value 1, and white moving first can go to 0. We have $\{1|0\} = \int\{0|1\} = \int\frac{1}{2}$. The labeled examples are taken from the Go literature; thus [M1, #37] means Problem 37 in [M1] (also called [Maeda 1965a]).

3.2. $1\frac{1}{2}\mathcal{E}$**.** We saw a few examples of $1\frac{1}{2}\mathcal{E}$ earlier. Many of the classical unsettled eyeshapes have this value, including all of the three- to six-point nakade shapes [Davies 1975, pp. 13–14, 22–27; Berlekamp and Wolfe 1994, p. 156], and "seven on the second line" [Segoe 1960, p. 10; Davies 1975, pp. 18–19]. It occurs frequently in the small closed and open corridors whose values are given later, and can also be constructed as the sum of two smaller eyespaces. See Figure 5.

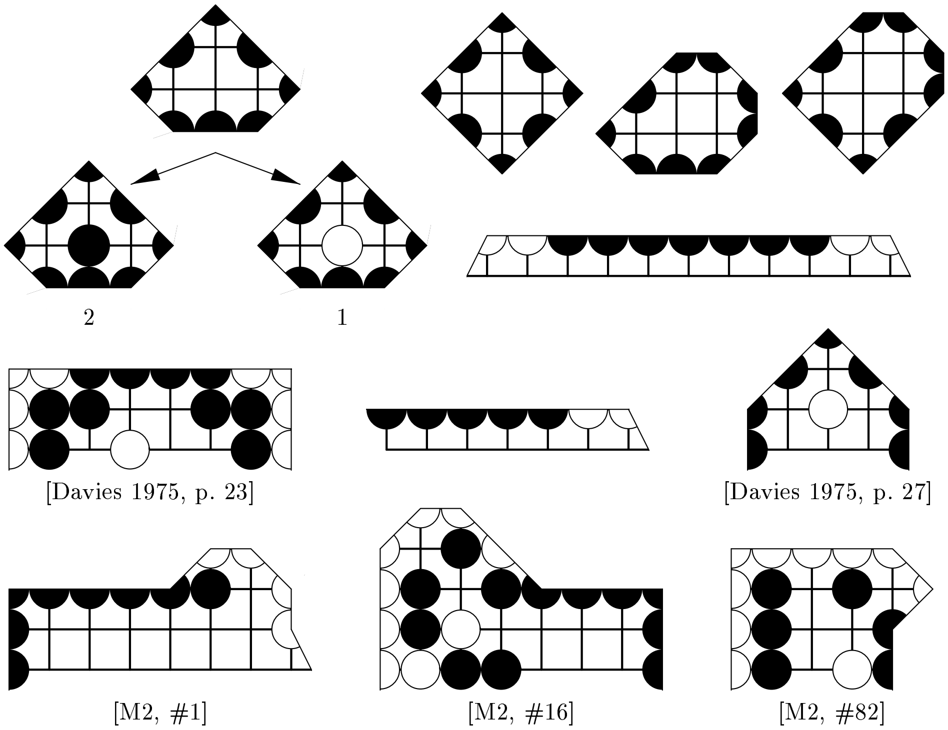


Figure 5. $\{2|1\} = \int\{1|2\} = \int 1\frac{1}{2}$.

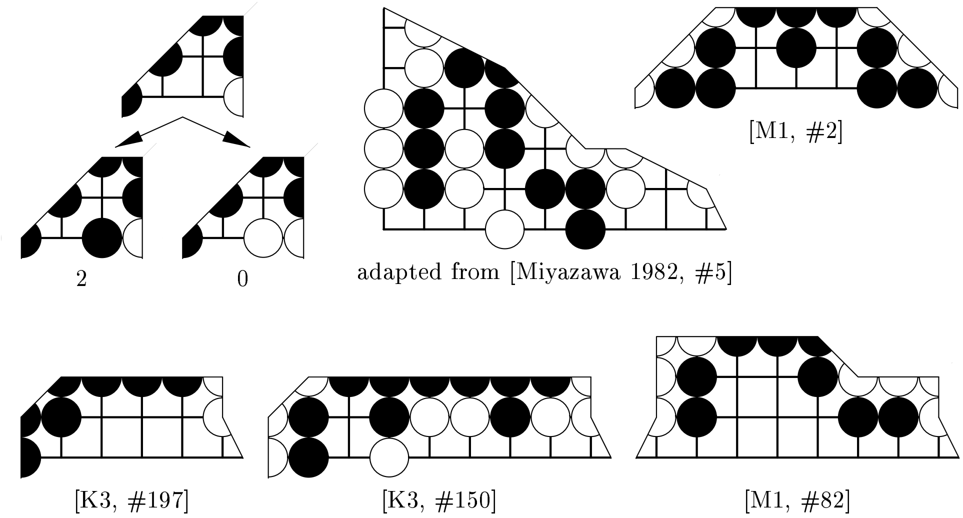


Figure 6. $\{2|0\} = \int\{1|1\} = \int 1*$.

3.3. $1{*}\mathcal{E}$**.** Both $\frac{1}{2}\mathcal{E}$ and $1\frac{1}{2}\mathcal{E}$ were simple gote games, what Yedwab [1985] calls switches. The largest possible switch is between two eyes and no eyes, and is written $1{*}\mathcal{E}$. See Figure 6.

3.4. $\frac{3}{4}\mathcal{E}$**.** When Black can make $1\frac{1}{2}\mathcal{E}$ in one move, it is better than only being able to make one eye in gote ($\frac{1}{2}\mathcal{E}$), and worse than being able to make two eyes in gote ($1{*}\mathcal{E}$). This game is $\frac{3}{4}\mathcal{E}$. In each of the examples, it is not enough for Black to guarantee one eye with his first move; he must also create an additional half eye. See Figure 7.

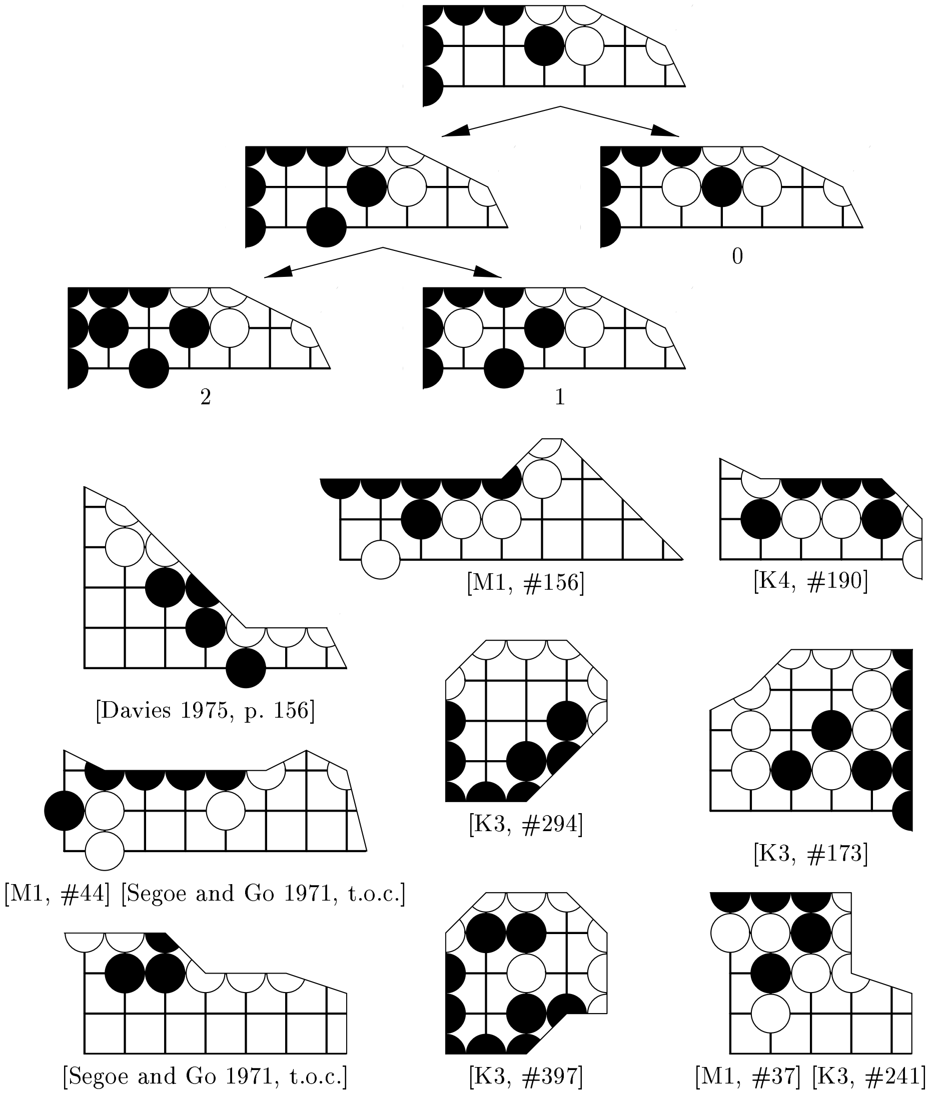


Figure 7. $\{\{2|1\}|0\} = \int \frac{3}{4}$. Here and in following figures t.o.c. stands for "table of contents".

3.5. $1\frac{1}{4}\mathcal{E}$**.** The complement of $\frac{3}{4}\mathcal{E}$ is $1\frac{1}{4}\mathcal{E}$. In this situation, Black can make two eyes in one move, but White can reduce him to only $\frac{1}{2}\mathcal{E}$. See Figure 8.

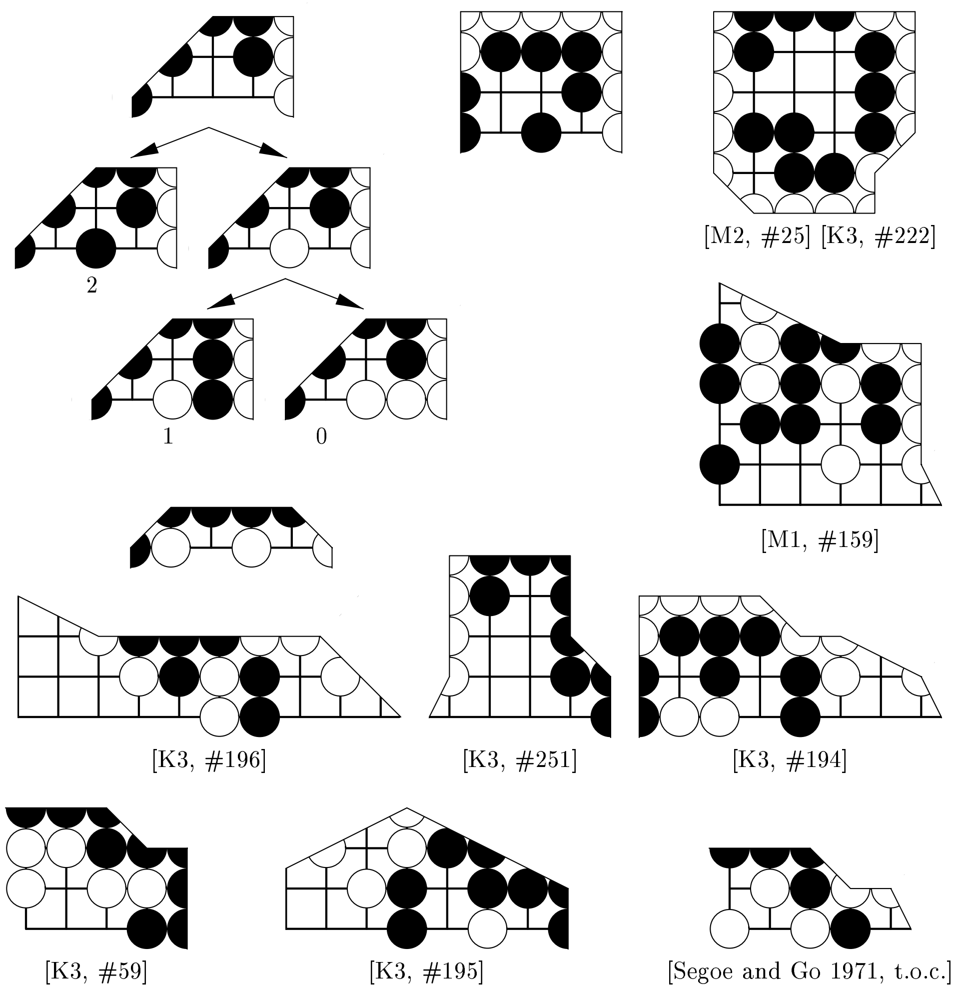


Figure 8. $\{2|\{1|0\}\} = \int 1\frac{1}{4}$.

3.6. Values of finite games in Bargo$_{[0,2]}$. Restricting the number-ish end-points of games to be one of 0, 1, or 2 is a fairly severe condition that (modulo infinitesimals) leaves only a few finite games with positive temperature, and a large class of infinite games of which only a fraction appear to be relevant to Go. The set of values we have seen so far form a sub-semigroup of Bargo$_{[0,2]}$. Table 1 shows the result of collapsed addition for these elements.

Early printings of [Berlekamp and Wolfe 1994] erroneously say on page 108 that $1 + \int \frac{3}{4} = \int 1\frac{1}{4}$; but $\int 1\frac{3}{4}$ actually collapses to $\int 1\frac{1}{2}$.

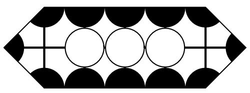
	0	$\int \frac{1}{2}$	$\int \frac{3}{4}$	$\int 1{*}$	1	$\int 1\frac{1}{4}$	$\int 1\frac{1}{2}$	2
0	0	$\int \frac{1}{2}$	$\int \frac{3}{4}$	$\int 1{*}$	1	$\int 1\frac{1}{4}$	$\int 1\frac{1}{2}$	2
$\int \frac{1}{2}$	$\int \frac{1}{2}$	1	$\int 1\frac{1}{4}$	$\int 1\frac{1}{4}$	$\int 1\frac{1}{2}$	$\int 1\frac{1}{2}$	2	2
$\int \frac{3}{4}$	$\int \frac{3}{4}$	$\int 1\frac{1}{4}$	$\int 1\frac{1}{2}$	$\int 1\frac{1}{2}$	$\int 1\frac{1}{2}$	2	2	2
$\int 1{*}$	$\int 1{*}$	$\int 1\frac{1}{4}$	$\int 1\frac{1}{2}$	2	$\int 1\frac{1}{2}$	2	2	2
1	1	$\int 1\frac{1}{2}$	$\int 1\frac{1}{2}$	$\int 1\frac{1}{2}$	2	2	2	2
$\int 1\frac{1}{4}$	$\int 1\frac{1}{4}$	$\int 1\frac{1}{2}$	2	2	2	2	2	2
$\int 1\frac{1}{2}$	$\int 1\frac{1}{2}$	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2

Table 1. Collapsed addition of values in $\mathrm{Bargo}_{[0,2]}$.

Note that Table 1 is taking some (infinitesimal) liberties with game arithmetic. For example, $\int \frac{1}{2} + \int \frac{3}{4} = \{1|0\}+\{\{2|1\}|0\} = \{2|1, \int \frac{1}{2}\}$, whereas we call it $\int 1\frac{1}{4} = \{2| \int \frac{1}{2}\}$. In normal game arithmetic, the move to 1 is not dominated by the move to $\int \frac{1}{2}$, because in some circumstances 1 is infinitesimally better (for White) than $\int \frac{1}{2}$. However, giving Black a sure eye (value 1) is always at least as bad for White as giving Black a chance to make an eye in gote $(\int \frac{1}{2})$. Ignoring the infinitesimals left after the number of eyes is decided makes some moves effectively dominated that would not otherwise be so. We can justify this sort of reduction by considering the same game cooled by a tiny amount $d > 0$. Then it is clear that 1 is always worse for White than $\{1-d|0+d\}$, no matter how small we make d, and regardless of any infinitesimal modifications to either game. So our method of simplifying game G can be viewed as mapping $G \mapsto G_{0+}$, the limit of G_δ (G cooled by δ) as δ approaches zero. This is equivalent to taking the *thermal dissociation* of G [Berlekamp et al. 1982, p. 164] and discarding the term of temperature zero, which contains all the infinitesimals.

Wolfe [Berlekamp and Wolfe 1994, p. 107] states that the only (unchilled) numbers that can occur are 0, 1, and 2, and that therefore the only chilled values that can occur in this context are 0, $\frac{1}{2}$, $\frac{3}{4}$, $1{*}$, 1, $1\frac{1}{4}$, $1\frac{1}{2}$, and 2. The next section, however, shows that this need not be the case.

3.7. Seki as an eyespace value. Sometimes an eyespace can be partly filled with enemy stones in such a way that neither player wants to play next, in one form of the situation known as *seki*:

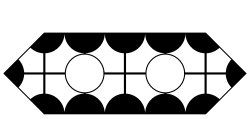


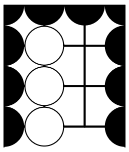
S

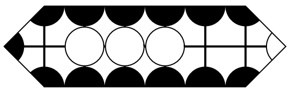
If Black tries to capture the White stones, he is left with only one eye; on the other hand, if White moves inside this eyeshape, Black can capture and gets 2 eyes. Formally this gives a simple seki-eye like this the value $S = \{1|2\} = 1\frac{1}{2}$. This is a number, not a hot game, and neither player wants to move first in it. Thus an eyeshape of value S is enough to prevent Black from being killed by White, even if Black has no other eyes or liberties.

Under some circumstances, living in seki may be almost as good as living with two eyes. We could construct a consistent theory in which seki is arbitrarily assigned the value 2, which would model this. That theory would then only have the finite eyespace values found in the previous section. However, there seems to be no reason (other than simplicity) to rob ourselves of the extra resolving power of treating living in seki as different from living with two eyes.

Once we allow S as a value, there are also games where one of the number-ish endpoints is S. They have values such as the following:

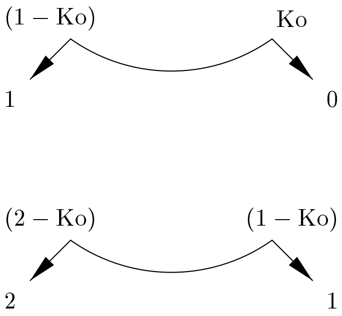

$2|S$

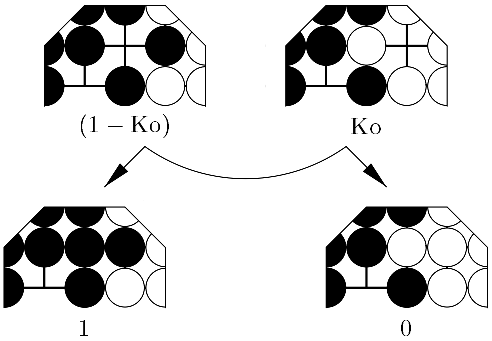

$S|1$


$S \mid \int \frac{1}{2}$

We have not determined the complete set of seki-related values, but it includes at least S, $2|S$, $S|1$, $S| \int \frac{1}{2}$, $S|0$, $2|S\| \int \frac{1}{2}$, $2\|S|1$, $2\|S| \int \frac{1}{2}$, $2\|S|0$, and $\{2| \int \frac{3}{4}, \{S|0\}\}$.

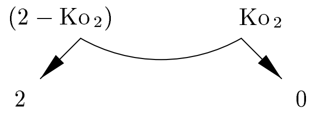
3.8. Ko, (1 − Ko), (1 + Ko), (2 − Ko). The cyclical situation known as *ko* gives rise to a number of loopy games. If we denote as Ko the value in a simple ko fight over 1 eye, where Black can take the ko and White can win the ko, the position after Black takes has value (1 − Ko). These values can also be translated by adding one eye to give (1 + Ko) and (2 − Ko).

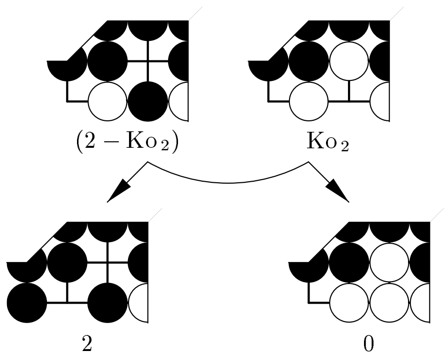




Since Ko + Ko + Ko = 1, we also have Ko + Ko = (1 − Ko) and $\mu(\text{Ko}) = \frac{1}{3}$. These games can also be viewed as the multiples of Ko: 1·Ko = Ko, 2·Ko = (1 − Ko), 3·Ko = 1, 4·Ko = (1 + Ko), 5·Ko = (2 − Ko), 6·Ko = 2.

3.9. $\mathbf{Ko_2}$ and $\mathbf{(2 - Ko_2)}$. It is also possible to have a ko fight over two eyes. The structure of the game is the same as for the one-point ko, but all the values are twice as large. Ko_2 has temperature and mean value of $\frac{2}{3}$.





3.10. Other loopy values. There are other loopy values that derive from simple ko fights. They include:

- *two-stage ko*, where winning the first stage ko advances Black to a second stage ko;
- *approach move ko* or *multi-step ko*, where one side must ignore ko threats to fill outside liberties;
- *mannen-ko* (10,000 year ko).

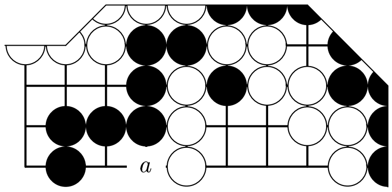
Examples are given in [Davies 1975, pp. 9–10]. Some loopy games in Go have periods that are longer than two moves, including:

- *chosei* or eternal life [Haruyama 1979], with period four. Under current Japanese (Nihon Kiin) rules, if chosei occurs the game is declared "no result" and must be played over; this has happened once [Nakayama 1989]. Under the generalized ko rule, chosei plays much like a normal ko: after three moves in the cycle, the fourth move is forbidden, so one player makes a ko threat. If the threat is answered, three more moves can be played in the cycle, and then it is the other player's turn to make a ko threat.
- *rotating ko* [Haruyama 1979], a loop of period eight, where either player has the option at certain positions to convert the game to a seki.

Given the bewildering variety of loopy games in Go hinted at by these last two examples, we cannot hope to provide an exhaustive categorization of their values, even for the single-group case. Indeed, Robson has shown that the family of life-and-death problems involving multiple simple kos is Exptime-complete [Robson 1981; 1982; 1983; 1985].

3.11. Connecting out. Sometimes a group can live, not by creating eyes within its "own" eyespaces, but by connecting out to another group of the same color. In most tsume-go problems, it is assumed that such a connection makes the group completely alive; this implies that we should treat it as being worth two eyes. This makes a connection in gote worth $1{*}\mathcal{E}$. A connection may also be made with another group that has G (less than two) eyes; such a potential connection in gote is worth $\{G\,|\,0\}$ eyes.

3.12. Limitations of Bargo$_{[0,2]}$ as a model for Go eyespace values. An eye in sente. We've seen that Bargo$_{[0,2]}$ is reasonably successful at modeling eyespace values in Go. However, it is not perfect. One defect is that, by limiting the maximum eye score to 2, we eliminate any possibility of a large threat in the process of making or destroying an eye. In particular, the games "an eye in sente" and "an eye that can be taken away in sente" have values that are outside of $\mathbf{Ug}_{[0,2]}$. Perhaps an example will make this clearer:



What is the value to the Black group in the corner of Black's eyespace around a? Within Bargo$_{[0,2]}$ the answer is $\frac{3}{4}\mathcal{E}$, since Black can make one eye while threatening to make another. But making the other eye kills the White group! In Bargo (without collapsing), killing the White group gets Black at least four more eyes, so the game is roughly $\{\{5\,|\,1\}\,|\,0\}$, which chills to $\{\{3\,|\,1\}\,|\,1\}$ or 1 plus miny-2, so the eyespace has a value something like $1 -_2 \mathcal{E}$. The precise value of this "eye in sente" depends on the size of the external threat; different threats give different (warmed) infinitesimals. However, all of these games collapse to $\frac{3}{4}\mathcal{E}$. The difference in external threats is lost.

We can also have an eye that can be taken away in sente, $\{1\,|\,\{0\,|\,{-x}\}$. In that case the pure game-theoretical value is a positive (warmed) infinitesimal that depends on x. But all of these collapse to $\frac{1}{2}\mathcal{E}$. Again the difference in threats is lost.

A serious problem arises when trying to add such values. Given the collapsed values above, one would expect the sum of an eye in sente and an eye that can be taken away in sente to be $\frac{1}{2}\mathcal{E} + \frac{3}{4}\mathcal{E} = 1\frac{1}{4}\mathcal{E}$. But $1\frac{1}{4}\mathcal{E}$ has Black stop two eyes and White stop one eye, whereas the actual result depends on which threat is greater. If White's threat is greater, the Black stop is only one eye, and the White stop zero eyes. Thus we see that some vital information can be lost through collapsing when large external threats are involved.

3.13. Values of open and closed corridors. Figures 9–11 give the Bargo$_{[0,2]}$ values for all closed corridors of length one to five and all open (at one end) corridors of length one to six. All of the non-ko-related, non-seki-related values described previously $\big(0,\ \textstyle\int \frac{1}{2},\ \int \frac{3}{4},\ 1,\ \int 1{*},\ \int 1\frac{1}{4},\ \int 1\frac{1}{2},\ 2\big)$ occur in this context.

Every closed corridor is worth at least one eye to Black; this makes 0, $\int \frac{1}{2}$, $\int \frac{3}{4}$, $\int 1*$, and $\int 1\frac{1}{4}$ impossible, since they all have a 0-ish endpoint.

If the Black stones surrounding an open corridor are all considered immortal, then the corridor is what Moews [Moews a] calls a *hyper-Black room*. In that case, the eye value of the corridor is irrelevant since the group has two eyes elsewhere, and the best point-making move for either player is at the mouth of the corridor (blocking for Black, pushing in for White). However, when the number of eyes matters, this is no longer so; in some corridors the move at the mouth is not optimal for either player.

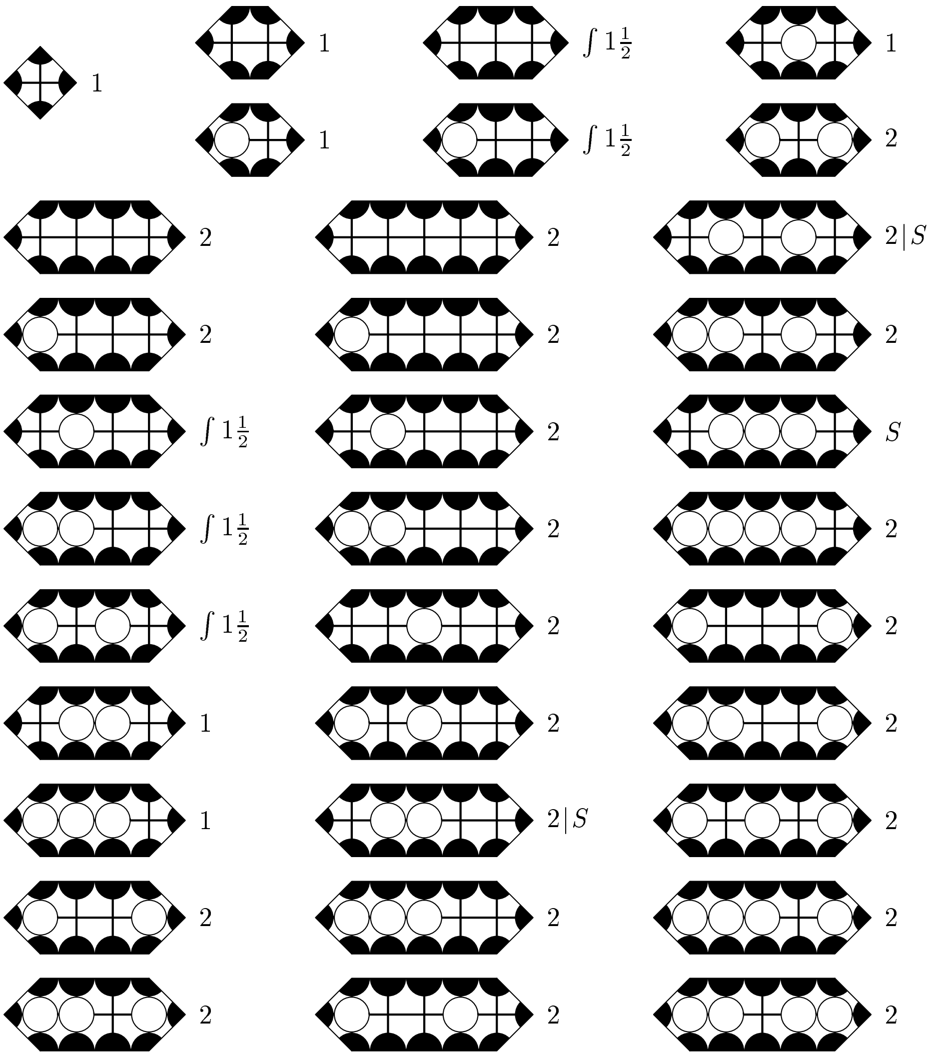


Figure 9. Eyes in closed corridors of length 1 to 5.

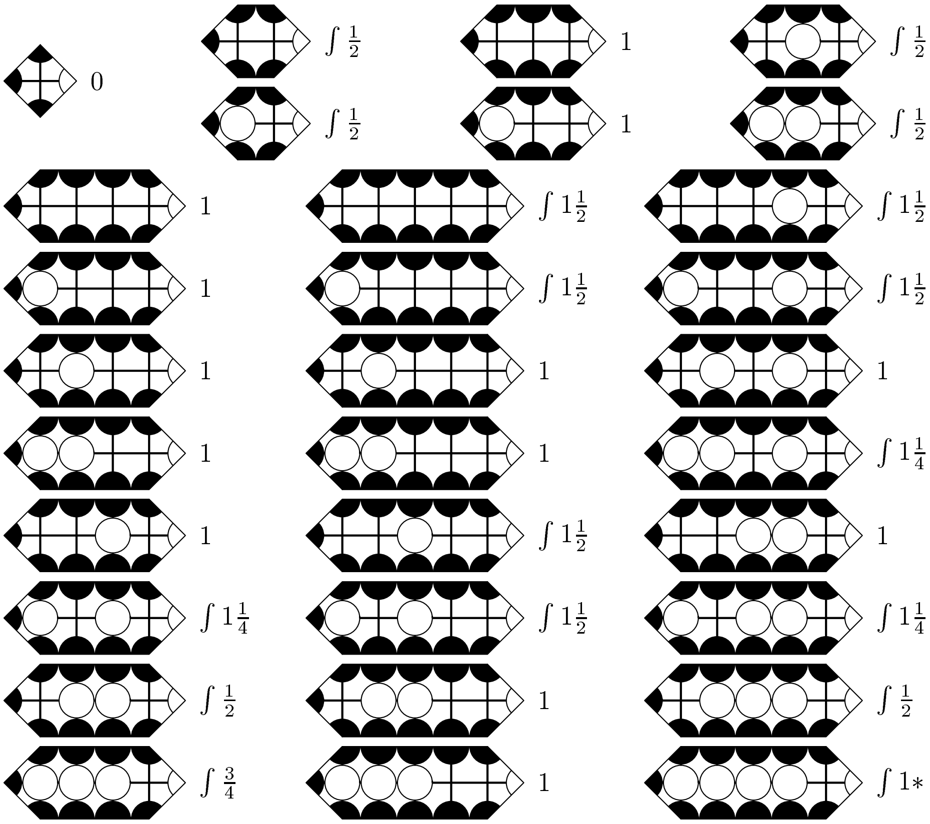


Figure 10. Eyes in open corridors of length 1 to 5.

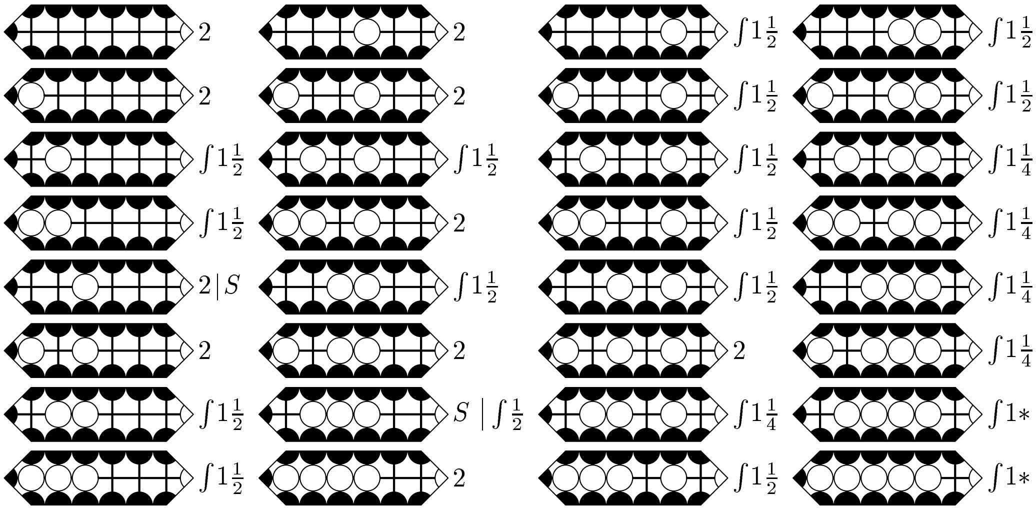
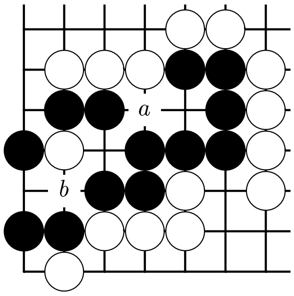


Figure 11. Eyes in open corridors of length 6.

4. A Mathematical Definition of Miai

One of the more subtle concepts of Go is *miai*, from *miru* (see, look at, observe) and *au* (meet, fit, be appropriate for). In non-Go Japanese, miai means "a marriage interview" or "an exchanging of glances". Nagahara devotes a chapter to miai [1972, pp. 3–4], and defines the Go sense of the word: "Miai means 'seeing together'. It refers to two points that are related in such a way that if one of them is occupied by a player, his opponent can handle the situation by taking the other." Davies [1975, p. 12] says that "Two points are miai if they represent two independent ways of accomplishing the same thing, so that if the enemy deprives you of one of them, you can always fall back on the other of them." As long as we view miai in terms of points or intersections, it is hard to apply game theory to it, for single intersections in general are not usually subgames, let alone independent ones. However, miai is also frequently applied to independent subgames, either in the context of scoring points, or in the context of making eyes. For example, Davies [1975, p. 71] describes a and b in the following position as miai:



A little analysis shows that the eyespaces around a and b (call them A and B) are independent of each other, and that A is $\frac{1}{2}\mathcal{E}$ and B is $1\frac{1}{2}\mathcal{E}$. Their sum is exactly two eyes. If White plays a, Black can live by playing b; and if White plays b, Black can live at a. Since $A + B$ is a number (in terms of eyes), the Number Avoidance Theorem [Berlekamp et al. 1982, pp. 144 and 179] tells us that neither player needs to move in this sum while there are still nonnumber games to be played. This lack of urgency is characteristic of miai. Nagahara writes "An important point to notice about miai is that the two moves involved are often not urgent. That is, they are in a state of equilibrium. ... it is not necessary for [Black] to rush to play either 'a' or 'b' since either point will give him life." It thus seems possible to formalize the concept of miai as follows:

Definition. A set of games is *miai* if none of them are numbers but their sum is a number.

The proviso that the sum not contain numbers prevents using miai to describe a single number-valued eyespace, which seems outside the spirit of the Go usage. In practice, it is possible to extend this definition slightly, and to call sums of unchilled games miai if they are infinitesimally close to a number. In that case, the chilled games will be miai by the above definition, since even a tiny amount of cooling makes the infinitesimals vanish.

Since any sum of miai is also miai, just knowing that a set of games is miai doesn't say much about the relationships among specific games in the set. We can address this by tightening the definition a bit:

Definition. A set of games is *irreducible miai* if it is miai and no proper subset of it is miai.

The irreducible miai in $\mathrm{Bargo}_{[0,2]}$ include (but are not limited to):

$$\begin{aligned}
\tfrac{1}{2}\mathcal{E} + \tfrac{1}{2}\mathcal{E} &= 1 \text{ eye}\\
\tfrac{1}{2}\mathcal{E} + 1\tfrac{1}{2}\mathcal{E} &= 2 \text{ eyes}\\
\tfrac{1}{2}\mathcal{E} + \tfrac{3}{4}\mathcal{E} + \tfrac{3}{4}\mathcal{E} &= 2 \text{ eyes}\\
\tfrac{3}{4}\mathcal{E} + 1\tfrac{1}{4}\mathcal{E} &= 2 \text{ eyes}\\
1{*}\mathcal{E} + 1{*}\mathcal{E} &= 2 \text{ eyes}\\
\mathrm{Ko}\mathcal{E} + \mathrm{Ko}\mathcal{E} + \mathrm{Ko}\mathcal{E} &= 1 \text{ eye}\\
\mathrm{Ko}\mathcal{E} + (1-\mathrm{Ko})\mathcal{E} &= 1 \text{ eye}\\
\mathrm{Ko}\mathcal{E} + (2-\mathrm{Ko})\mathcal{E} &= 2 \text{ eyes}\\
(1+\mathrm{Ko})\mathcal{E} + (1-\mathrm{Ko})\mathcal{E} &= 2 \text{ eyes}\\
\mathrm{Ko}_2\mathcal{E} + \mathrm{Ko}_2\mathcal{E} + \mathrm{Ko}_2\mathcal{E} &= 2 \text{ eyes}\\
\mathrm{Ko}_2\mathcal{E} + (2-\mathrm{Ko}_2)\mathcal{E} &= 2 \text{ eyes}
\end{aligned}$$

The above only includes miai that are exact without collapsing, and hence are miai in Bargo as well as $\mathrm{Bargo}_{[0,2]}$. There are others that do not meet this criterion, such as:

$$1\tfrac{1}{2}\mathcal{E} \oplus 1{*}\mathcal{E} = c(2\tfrac{1}{2}{*}\mathcal{E}) = 2 \text{ eyes}.$$

The sum of any (noninteger) game and its complement is necessarily a miai for two eyes.

One surprising consequence of the above definitions is that a set of three or more games may be miai even though no subset of them is! This possibility has not, to my knowledge, been considered by professional Go players or writers; this may be because ai is usually used to indicate a relationship between two objects.

Since the temperature of a number is less than the temperature of any non-number (except for infinitesimals, which we are ignoring anyway), miai involves a kind of mutual cancellation of temperature. The temperature of the sum is less than the temperature of any summand. Such cancellation may occur more weakly than in miai:

Definition. A set of games is *partial miai* if it is not miai but the sum of the games has lower temperature than any game in the set.

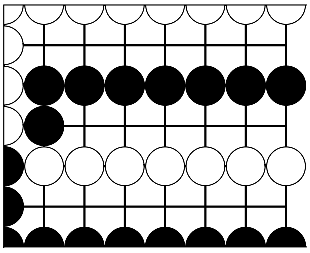
We can also define *irreducible partial miai*; an example is $\frac{3}{4}\mathcal{E} + \frac{3}{4}\mathcal{E} = 1\frac{1}{2}\mathcal{E}$.

5. Semeai: Two-group Life and Death

5.1. Semeai. Sometimes life and death problems in Go involve more than a single group at risk. One common situation is called *semeai*, defined by Bauer [Segoe 1960, p. 6] as "a localized situation where only one of the two opposing groups of stones can live (unless the result is a seki), and therefore each must try to kill the other without any reference to or connection with other stones on the board".

To analyze semeai we must take account of both Black and White eyes, which will be the motivation for introducing the game Argo below. However, before proceeding, it is necessary to discuss the manner in which eyes and liberties affect the outcome of a semeai.

5.2. Semeai where no eyes are possible. Before considering the role of eyemaking moves in a semeai, it will help to understand the values of semeai where the eyes have already been decided. The simplest case is one where neither Black nor White can make any eyes, there are no kos, and all liberties are simple dame. In this case, life or death is entirely a function of liberty count.



If the number of shared liberties is less than two, seki is not possible; either the Black group or the White group must die. This means that there are no zero games, and all values are either positive or negative or fuzzy. Whichever group has the most outside liberties wins, or if the outside liberties are equal then the player moving first wins. Letting BOL and WOL stand for the number of Black outside liberties and White outside liberties respectively, we can summarize the above by saying that the game's outcome class is the same as that of the game $(\mathrm{BOL} - \mathrm{WOL} + *)$. Whether the number of shared liberties is zero or one makes no difference. If the number of shared liberties is two or more, then seki is a conceivable result. For Black to capture White, Black must fill all White's outside liberties as well as all the shared liberties. Black must also have at least one outside liberty remaining when he fills the next-to-last shared liberty, else White will capture him.

Table 2 summarizes this in formulas and in a grid. We can see that the result space is partitioned into three large cool areas (W, S, and L), with thin hot boundaries (WS, WL, and SL) between them. Cool cells adjacent to the boundaries imply that there are ko threats for the side that is one move away

Black wins if (BOL − WOL) $\geq$ Shared.
Black wins moving first if (BOL − WOL) $\geq$ Shared − 1.
White wins if (WOL − BOL) $\geq$ Shared.
White wins moving first if (WOL − BOL) $\geq$ Shared − 1.
If neither Black nor White wins, the result is seki.

		BOL − WOL										
		5	4	3	2	1	0	−1	−2	−3	−4	−5
Shared Liberties	5	W	WS	S	S	S	S	S	S	S	SL	L
	4	W	W	WS	S	S	S	S	S	SL	L	L
	3	W	W	W	WS	S	S	S	SL	L	L	L
	2	W	W	W	W	WS	S	SL	L	L	L	L
	1	W	W	W	W	W	WL	L	L	L	L	L
	0	W	W	W	W	W	WL	L	L	L	L	L

Table 2. Semeai outcomes when no eyes are possible. BOL = Black outside liberties; WOL = White outside liberties; W = Black wins, White loses; S = Seki; L = Black loses, White wins; WL = $\{W|L\}$; WS = $\{W|S\}$; SL = $\{S|L\}$.

from being able to improve its status. For example, Black filling a White outside liberty and moving left from an L cell to an SL cell threatens to make a seki; White must answer (either by filling a Black outside liberty and moving right, or, if Black has no outside liberties, by filling a shared liberty and moving down) to return to L status and keep Black dead. The S cell with two shared liberties and (BOL − WOL) = 0 is unique in that *both* sides have ko threats.

5.3. Semeai where each side has one eye. When each side in a semeai has one single-point eye, the situation is similar to that for no eyes. The pair of opposing eyes behaves somewhat like a single shared liberty. The results are shown in Table 3.

Note that, for a seki to be possible between two one-eyed groups, they must also have at least one shared liberty.

		BOL − WOL										
		5	4	3	2	1	0	−1	−2	−3	−4	−5
Shared Liberties	5	WS	S	S	S	S	S	S	S	S	S	SL
	4	W	WS	S	S	S	S	S	S	S	SL	L
	3	W	W	WS	S	S	S	S	S	SL	L	L
	2	W	W	W	WS	S	S	S	SL	L	L	L
	1	W	W	W	W	WS	S	SL	L	L	L	L
	0	W	W	W	W	W	WL	L	L	L	L	L

Table 3. Semeai outcomes when each side has one eye.

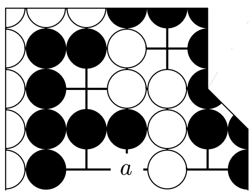
	BOL − WOL										
	5	4	3	2	1	0	−1	−2	−3	−4	−5
Shared Liberties 5	W	W	W	W	W	W	W	W	W	W	W
4	W	W	W	W	W	W	W	W	W	W	WL
3	W	W	W	W	W	W	W	W	W	WL	L
2	W	W	W	W	W	W	W	W	WL	L	L
1	W	W	W	W	W	W	W	WL	L	L	L
0	W	W	W	W	W	W	WL	L	L	L	L

Table 4. Semeai outcomes when black has one eye and white has none.

5.4. Semeai with unbalanced eyes. We have already looked at cases with 0 versus 0 eyes and 1 versus 1 eyes. Semeai where one group has two eyes are not semeai at all; the group with two eyes is alive, and if the other group cannot make two eyes it is simply dead, regardless of liberty count.

The only remaining possibility with integer eyes is 1 eye versus 0 eyes. As in all two-group semeai with unequal number of eyes, seki is not possible; one group or the other must die. For the case where Black has one eye and White has none, the results are shown in Table 4. The case where White has one eye and Black has none is symmetric.

To a first approximation, the advantage of one eye converts the "seki region" in the previous tables into wins for Black. This is so beneficial that one might be tempted to infer that it is always better to make an eye than to worry about liberties. In fact there is a Go proverb that states "The semeai where only one player has an eye is a fight over nothing" [Segoe 1960, p. 76]. However, making an eye sometimes uses up more than one liberty (one converted to the eye, and one used up by the play), as well as taking a move (which might otherwise be used to fill an enemy liberty). In the borderline cases, there are counterexamples to this generalization.



In this example, if Black makes an eye at a White will have enough outside liberties to kill him. The locally correct line of play is for Black to fill one of White's outside liberties, which forces White to play a, and then Black fills the other outside liberty to leave a seki. If Black had even one more liberty, either outside or shared, then the locally correct line of play would be to make the eye, which kills White. Segoe [1960, p. 79] observes: "Even in those cases where one player has one eye and the other has none the number of dame available to each player must be carefully analyzed or the semeai may be lost."

This interaction between eyes and liberties is awkward, and a full analysis of it is beyond the scope of the present paper. Such an analysis might also require, as a prerequisite, a better understanding of the subgames of Go where making or destroying liberties is the main objective, which seems to be a fairly rich topic in its own right. However, if the number of shared liberties is sufficiently greater than the absolute value of BOL − WOL, these complications do not occur. In order to simplify the analysis in what follows, we will usually assume that the number of shared liberties is sufficiently large that the outside liberty count can be ignored, so that whichever side makes the most eyes wins the battle or, if the eyes made are equal and less than two, both sides live in seki. Table 5 shows the possible results of a semeai as a function of the number of Black and White eyes, in the fully general case and after the simplifying assumption has been made. After this simplification, the result depends only on the difference between the number of Black eyes and White eyes.

		White eyes 0	1	≥ 2
Black eyes	≥ 2	W	W	both live
	1	W, L	W, S, L	L
	0	W, S, L	W, L	L

		White eyes 0	1	≥ 2
Black eyes	≥ 2	W	W	both live
	1	W*	S*	L
	0	S*	L*	L

Table 5. Possible results as a function of the number of eyes, under no assumptions (left) and under the assumption of enough shared liberties (right). Stars mark cases that depend on the assumption.

5.5. Modeling Semeai: Argo. When two groups are locked together in a semeai, the one to make the most eyes usually wins (as we have seen above). Under appropriately restrictive conditions (neither group can make more than two eyes, it is not possible for both groups to make two eyes, and there are enough shared liberties), these struggles can be solved simply by counting eyes, with the value of each eyespace being a game whose integer endpoints are between -2 and 2. Since both White and Black can make eyes in such a semeai, the natural score is the number of Black eyes minus the number of White eyes. If this is ≥ 1, Black wins; if it's ≤ -1, White wins; if it's 0, both groups are alive in seki; otherwise, the result depends on who moves first.

To model this we define a Go-like game that I call Argo (after Argus Panoptes, who had a hundred eyes; the ship Argo which the Argonauts sailed was named after a different Argus, son of Phrixus and quite human). The rules of Argo are the same as Go, with the exception of scoring: in Argo Black and White each get one point for every eye they have at the end of the game. That is, we count each Black eye as $+1$ and each White eye as -1.

In the case of Bargo (whose name, we can now reveal, stands for Black Argo), we wanted to restrict all values to have integer endpoints between 0 and 2 inclusive. For Argo, the natural limits are -2 and 2.

Using $\text{Argo}_{[-2,2]}$ we can model situations where both Black and White eyes can be made. All of the values from $\text{Bargo}_{[0,2]}$, their negatives, and many additional values such as $\frac{1}{4}\mathcal{E}$ and $*\mathcal{E}$ occur. Note that some of the games given above as examples for $\text{Bargo}_{[0,2]}$ have different values in $\text{Argo}_{[-2,2]}$, since the White eyes now count!

However, something seems not quite right. If we represent a result where Black has b eyes and White has w eyes by the ordered pair (b, w), the central problem is that there are real differences between $(0, 0)$, $(1, 1)$, and $(2, 2)$ when it comes to adding them to other games; but, they are all represented in Argo as the game 0. And $(0, 0) + G$ is always equal to G, but $(2, 2) + G$ is equal to $(2, 2)$ regardless of G.

The conclusion seems to be that adding negative White eyes to positive Black eyes is too simplistic; we really need to treat each separately (clipping against limits of 0 and 2). But that implies that the "integer" endpoints of each game need to be represented as an ordered pair of integers (x, y), or as a complex integer $x + iy$. Accepting that means that we must go outside of the established theoretical framework and construct a combinatorial game theory of complex- or (more generally) vector-valued games.

6. Directions for Future Research

6.1. Completely characterizing single-group seki-related values. The set of seki-related values in $\text{Bargo}_{[0,2]}$ is probably finite, and not much larger than the set of values mentioned earlier. It should be possible to identify all elements of this set and provide examples of each.

The asymmetry introduced by S as a value could be eliminated if a Go position was found that had value $\{0|1\} = \frac{1}{2}$ eyes. I do not know of any examples of this value. However, there are several known Go positions other than seki that have the property that neither player wants to move first, such as "three points without capturing" [Berlekamp and Wolfe 1994, pp. 165–168], so its existence does not seem impossible.

6.2. Characterizing loopy game values. Even for the single-group case, there are clearly many loopy games that still need to be identified and their values characterized. The same holds true for multi-group problems. There does not appear to be much hope of a general solution for all loopy games in Go, since already, as mentioned earlier, the family of life-and-death problems involving multiple simple kos is Exptime-complete [Robson 1981; 1982; 1983; 1985]. Robson's construction involves extremely complicated topologies that are unlikely to occur in real games, however. It may still be possible to understand the most common kos, which all have simple topologies.

6.3. Counting liberties. Since the number of liberties of a unit or group often has a critical effect on its survival, we can apply the theory to games in which making or eliminating liberties is the main goal. Liberties can be gained by connection, so the study of cut-or-connect problems will be relevant. Unlike eye-counting, liberty-counting has no built-in upper limit, so we should expect much hotter games to appear. In fact, making two eyes effectively supplies an infinite number (**on**) of liberties, so infinitely-hot games would not be unreasonable.

A firm theory of liberties, plus the vector-valued game theory mentioned earlier, might allow accurate analysis of situations where both eye-count and liberty-count are critical.

6.4. Ko threats. The existing theory, from [Conway 1976] to the present work, has the unfortunate effect of simplifying away ko threats. For example, if Black is topologically alive with two eyes, White has no threats against the Black group, but if Black is alive with exactly 1 eye + $\frac{1}{2}\mathcal{E}$ + $\frac{1}{2}\mathcal{E}$, White has at least one ko threat against Black (by moving one of the $\frac{1}{2}\mathcal{E}$ subgames to 0). Yet we treat both of these situations as identical, calling them two eyes.

Getting beyond this limitation would seem to require changes to the very foundations of the theory, altering the definition of equality and restricting the simplifications allowed.

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